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CHARACTERISTIC IMPEDANCES OF GENERALIZED STRIP-TRANSMISSION LINES

by

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INTRODUCTION

In order to study the effects of layout on high-speed circuits it is necessary to be able to calculate the characteristic impedance of a rectangular conductor placed parallel to, but spaced arbitrarily from, two ground planes. The situation is described in Fig. 1.

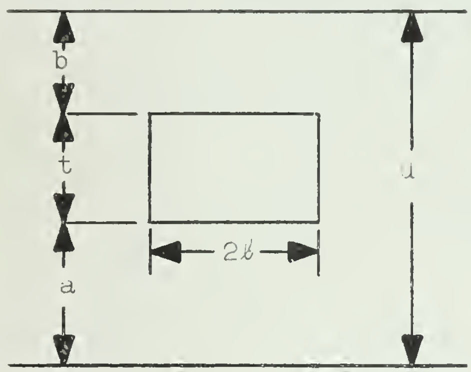

Figure 1. Generalized Strip Line

The difficulties with this type of geometry, as far as solutions to Maxwell's equations are concerned, is well known. However, approximate methods may be used to calculate an upper and a lower bounds to the characteristic impedance. The usefulness of the analysis is thus measurable.

1. THE LOWER BOUND

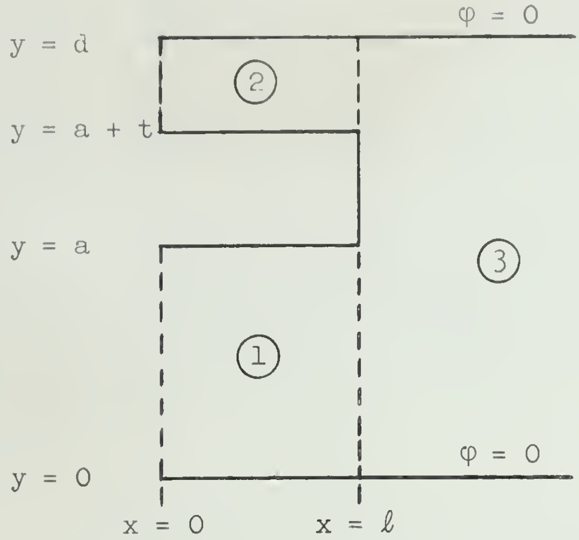

Figure 2. Boundary Values and Definition of Regions

The potentials in the three regions are given by:

$$\varphi_1 = \frac{V_0 y}{a} + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \tag{1.1}$$

$$\varphi_2 = \frac{V_0(y - d)}{b} + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi(d - y)}{b} \tag{1.2}$$

$$\varphi_3 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi y}{d} \epsilon^{-\frac{n\pi x}{d}} \tag{1.3}$$

since the potenials satisfy Laplace's equation $\nabla^2\varphi = 0$ and the boundary conditions. They are the solutions to the problem of Fig. 2. The continuity conditions are

$$\begin{aligned} \varphi_2(\ell,y) &= \varphi_2(\ell,y) \qquad a + t \leq y \leq d \\ \varphi_1(\ell,y) &= \varphi_2(\ell,y) \qquad 0 \leq y \leq a \end{aligned} \tag{1.4}$$

and

$$\frac{\partial\varphi_2}{\partial x}(\ell,y) = \frac{\partial\varphi_3}{\partial x}(\ell,y) \qquad a + t \leq y \leq d$$

$$\frac{\partial\varphi_1}{\partial x}(\ell,y) = \frac{\partial\varphi_3}{\partial x}(\ell,y) \qquad 0 \leq y \leq a \tag{1.5}$$

These equations may be used to evaluate the coefficients in the potentials. However, this method will not be used here. Instead an argument based on energy considerations will be used. The electrostatic energy between two conducting surfaces is given by Eq. (1.6).

$$W_e = \frac{1}{2}\,\epsilon \iint \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dxdy = \frac{1}{2}\, CV_0^2 \tag{1.6}$$

C = capacity per unit length

$$V_0 = \varphi_2 - \varphi_1$$

If the first-order variation in W_e due to a first-order variation in the functional form φ, where φ satisfies the boundary conditions, is computed, it is found that the variation vanishes if φ satisfies Laplace's equation; a condition which is already satisfied. Then

$$\frac{C}{\sqrt{\mu\epsilon}} = \frac{1}{Z_0^L} = \frac{\iint \bar{\nabla}_t\varphi \cdot \bar{\nabla}_t\varphi dxdy}{\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} V_0^2} \tag{1.7}$$

The integral is positive so that the stationary value is a minimum. It follows that

$$Z_0^L \leq Z_0^{exact} \tag{1.8}$$

The total energy is the sum of the energies in the separate regions:

$$W_1 = \frac{1}{2} \epsilon \sum_{n=1}^{\infty} \left[\iint \left(a_n \frac{n\pi}{a}\right)^2 \left(\sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}\right)^2 dxdy \right. \tag{1.9}$$

$$\left. + \iint \left[\frac{V_0}{a} + \left(a_n \frac{n\pi}{a}\right) \left(\cosh \frac{n\pi x}{a} \cos \frac{n\pi y}{a}\right) \right]^2 dxdy \right]$$

where

$$0 \leq x \leq \ell \qquad 0 \leq y \leq a$$

This may be integrated to yield:

$$W_1 = \frac{1}{2} \epsilon \left[\frac{V_0^2 \ell}{a} + \frac{\pi}{4} \sum_{n=1}^{\infty} a_n^2 n \sinh \frac{2n\pi\ell}{a} \right] \tag{1.10}$$

The energy in region 2 may be obtained from Eq. (1.10) by replacing a by b.

$$W_2 = \frac{1}{2} \epsilon \left[\frac{V_0^2 \ell}{b} + \frac{\pi}{4} \sum_{n=1}^{\infty} b_n^2 n \sinh \frac{2n\pi\ell}{b} \right] \tag{1.11}$$

In region 3:

$$W_3 = \frac{1}{2} \epsilon \sum_{1}^{\infty} \left(\frac{n\pi}{d}\right)^2 C_n^2 \int_0^d \int_\ell^\infty \left(\cos^2 \frac{n\pi y}{d} + \sin^2 \frac{n\pi y}{d}\right) \epsilon^{-\frac{2n\pi x}{d}} dxdy$$

$$W_3 = \frac{1}{2} \epsilon \left(\frac{n\pi}{2} C_n^2 \epsilon^{-\frac{2n\pi\ell}{d}} \right) \tag{1.12}$$

The total energy is thus:

$$W_T = \frac{1}{2} \epsilon \left[\frac{V_0^2 \ell}{b} + \frac{V_0^2 \ell}{a} + \frac{\pi}{4} \sum_{n=1}^{\infty} n \left(a_n^2 \sinh \frac{2n\pi\ell}{a} + b_n^2 \sinh \frac{2n\pi\ell}{b} + 2C_n^2 \epsilon^{-\frac{2n\pi\ell}{d}} \right) \right] \tag{1.13}$$

The coefficients may be evaluated in terms of the unknown potential distribution at $x = \ell$. The distribution is assumed to have the form:

$$G(y) = \begin{cases} V_0 \dfrac{y}{a} + g(y) & 0 \le y \le a \\ V_0 & 0 \le y \le t \\ V_0 \dfrac{d - y}{b} + f(y) & d - b \le y \le d \end{cases} \tag{1.14}$$

The equating of Eq. (1.14) to respective potentials at $x = \ell$ yields:

$$a_n \cosh \frac{n\pi\ell}{a} = \frac{2}{a} \int_0^a g(y) \sin \frac{n\pi y}{a}\, dy \tag{1.15}$$

$$c_n \epsilon^{-\frac{n\pi\ell}{d}} = \frac{2}{d} \int_0^d G(y) \sin \frac{n\pi y}{d}\, dy \tag{1.16}$$

$$b_n \cosh \frac{n\pi\ell}{d} = \frac{2}{b} \int_{d-b}^b f(y) \sin \frac{n\pi(d - y)}{b}\, dy \tag{1.17}$$

The approximations are introduced at this point. It will be assumed that to a first approximation $g(y) = f(y) = 0$. In this case only c_n has a nonzero value.

$$c_n \epsilon^{-\frac{n\pi\ell}{d}} = \frac{2}{d} \left[\int_a^{d-b} V_0 \sin \frac{n\pi y}{d}\, dy + \int_0^a V_0 \frac{y}{a} \sin \frac{n\pi y}{d}\, dy + \int_{d-b}^d V_0 \frac{d - y}{b} \sin \frac{n\pi y}{d}\, dy \right] \tag{1.18}$$

This may be integrated by standard methods to yield

$$c_n = \frac{2V_0 d}{n^2 \pi^2 a} \left(\sin \frac{n\pi a}{d} + \frac{a}{b} \sin \frac{n\pi(d - b)}{d} \right) \epsilon^{\frac{n\pi\ell}{d}} \tag{1.19}$$

The total energy becomes then:

$$W_T = \frac{1}{2} \epsilon V_0^2 \left[\frac{\ell}{b} + \frac{\ell}{a} + \frac{2d^2}{\pi^3 a^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sin \frac{n\pi a}{d} + \frac{a}{b} \sin \frac{n\pi(d - b)}{d} \right)^2 \right] \tag{1.20}$$

This is the total energy in the three regions of Fig. 2. The total energy in the entire cross-section is thus twice that amount.

Hence:

$$C = \frac{4W_T}{V_0^2}$$

$$Z_0^L = \frac{V_0^2 \sqrt{\mu\epsilon}}{4W_T}$$

or:

$$Z_0^L = \frac{Z_0}{2\kappa \left(\frac{\ell}{b} + \frac{\ell}{a} + \frac{2d^2}{\pi^3 a^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sin \frac{n\pi a}{d} + (-1)^{n+1} \frac{a}{b} \sin \frac{n\pi b}{d} \right)^2 \right)} \tag{1.21}$$

where

$$\epsilon = \epsilon_0 \kappa$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Two limiting cases are of interest.

(a) Symmetric: a = b

$$Z_0^L = \frac{Z_0}{4\kappa\left(\frac{\ell}{a} + \frac{4d^2}{\pi^3 a^2} \sum\limits_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin^2 \frac{n\pi a}{d}\right)} \tag{1.21a}$$

(b) Thin center conductor: t = 0, d = a + b

$$Z_0^L = \frac{Z_0}{2\kappa\left(\frac{\ell}{a} + \frac{\ell}{b} + \frac{2d^4}{\pi^3 a^2 b^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^3} \sin^2 \frac{n\pi a}{d}\right)} \tag{1.21b}$$

2. THE UPPER BOUND

It is possible to calculate an upper bound of Z_0 by making use of the Green's function approach.

The removal of a charged conductor from a system of conductors is permitted if the conductor is replaced by an equivalent surface charge.

$$\rho_s = \epsilon E_n = -\epsilon \frac{\partial \varphi}{\partial n}$$

The potential function which has to be found is then the solution of

$$\nabla_t^2 \varphi = - \frac{1}{\epsilon} \rho(x',y')$$

where, for the problem at hand:

$$\varphi(S_1) = 0 \qquad\qquad \rho(S_2) = \rho(x',y')$$

This problem may now be solved in terms of a unit line charge at position (x',y') in the presence of S_1. Thus, if

$$\nabla_t^2 \varphi(x,y) = - \frac{1}{\epsilon} \delta(x - x')\delta(y - y') \qquad\qquad \varphi(S_1) = 0$$

has the solution $G(x,y|x',y')$, then the potential due to the charge distribution $\rho(x',y')$ is

$$\varphi(x,y) = \oint_{S_2} G(x,y|x',y')\rho(x',y')d\ell'$$

The charge distribution is determined from the following equation, where use was made of the fact that $\varphi(x(S_2),y(S_2)) = V_0$.

$$V_0 = \oint_{S_2} G(x(S_2),y(S_2)|x',y')\,\rho(x',y')d\ell'$$

but

$$V_0 Q = V_0 \oint_{S_2} \rho(x,y)d\ell = \frac{Q^2}{C} = \oint_{S_2}\oint_{S_2} G(x(S_2),y(S_2)|x',y')\rho(x,y)\rho(x',y')d\ell d\ell'$$

Hence:

$$\frac{1}{C} = \frac{\oint_{S_2}\oint_{S_2} G(x,y|x',y')\rho(x,y)\rho(x',y')d\ell d\ell'}{\left[\oint_{S_2} \rho(x,y)d\ell\right]^2}$$

This expression is stationary for arbitrary first-order changes in the function $\rho(x,y)$. It is a minimum and therefore yields an upper bound on Z_0. Hence

$$Z_0^U \geq Z_0^{exact}$$

In order to use this method the Green's function for a line charge between conducting ground planes must be available. It is the solution of

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\frac{1}{\epsilon}\delta(x - x')\delta(y - y') \qquad G(x,0) = G(x,d) = 0 \qquad (2.1)$$

Since, if $x \neq x'$, $y \neq y'$, G must satisfy Laplace's equation, which is separable in rectangular coordinates, G must have the form:

$$G = \sum_n f_n(x)f_n(x')g_n(y)g_n(y') \qquad (2.2)$$

The y part may be determined by expanding the function $\delta(y - y')$:

$$\delta(y - y') = \sum_n a_n \sin \frac{n\pi y}{d} \tag{2.3}$$

$$\delta(y - y') = \frac{2}{d} \sum_n \sin \frac{n\pi y'}{d} \sin \frac{n\pi y}{d}$$

Hence, if $g_n(y)g_n(y') = \sin \frac{n\pi y'}{d} \sin \frac{n\pi y}{d}$, the differential equation for the x dependence becomes:

$$f_n(x') \left(\frac{d^2 f_n(x)}{dx^2} - \left(\frac{n\pi}{d}\right)^2 f_n(x) \quad = - \frac{2}{\epsilon d} \delta(x - x') \right) \tag{2.4}$$

Since G is a potential it is continuous at $x = x'$, $y = y'$. The discontinuity in the derivative is found by integrating Eq. (2.4) about $x' \pm \Delta$. This yields

$$\frac{df_n(x^+)}{dx} - \frac{df_n(x^-)}{dx} = - \frac{2}{f_n(x')\epsilon d} \tag{2.5}$$

Since G must remain finite as $x \to \infty$ suitable solutions are

$$f_n(x) = \begin{cases} a_n \epsilon^{\frac{n\pi x}{d}} & x < x' \\ b_n \epsilon^{-\frac{n\pi d}{d}} & x > x' \end{cases} \tag{2.6}$$

The coefficients may be obtained from Eq. (2.5) and the continuity of G. This yields, after some manipulations:

$$G = \begin{cases} \frac{1}{\pi\epsilon} \sum_n \frac{1}{n} \sin \frac{n\pi y'}{d} \sin \frac{n\pi y}{d} \epsilon^{\frac{n\pi}{d}(x-x')} & x < x' \\ \frac{1}{\pi\epsilon} \sum_n \frac{1}{n} \sin \frac{n\pi y'}{d} \sin \frac{n\pi y}{d} \epsilon^{\frac{n\pi}{d}(x'-x)} & x > x' \end{cases} \tag{2.7}$$

As a first approximation it will be assumed that the function ρ is given by

$$\rho(x,y) = \text{constant} \tag{2.8}$$

Since the quantity $\frac{1}{C}$ depends only on the functional form of ρ it is permissible to let the constant be unity. The denominator of $\frac{1}{C}$ becomes thus simply:

$$\left(\oint_{S_2} \rho(x,y) d\ell \right)^2 = 4(t + 2\ell)^2 \tag{2.9}$$

The problem is then reduced to the calculation of

$$J = \oint_{S_2} \oint_{S_2} G(x,y|x',y') d\ell d\ell' \tag{2.10}$$

The primed integration yields:

$$J_1 = \sin\frac{n\pi y}{d} \int_{-\ell}^{x} \left(\sin\frac{n\pi a}{d} + \sin\frac{n\pi(d-b)}{d} \right) \epsilon^{-\frac{n\pi}{d}(x-x')} dx'$$

$$+ \sin\frac{n\pi\ell}{d} \int_{x}^{\ell} \left(\sin\frac{n\pi a}{d} + \sin\frac{n\pi(d-b)}{d} \right) \epsilon^{\frac{n\pi}{d}(x-x')} dx'$$

$$+ \int_{a}^{d-b} \sin\frac{n\pi y}{d} \sin\frac{n\pi y'}{d} \left(\epsilon^{\frac{n\pi}{d}(x-\ell)} + \epsilon^{-\frac{n\pi}{d}(x+\ell)} \right) dy' \tag{2.11}$$

This may be integrated to yield:

$$J_1 = \frac{4d}{n\pi} \sin \frac{n\pi y}{d} \sin \frac{n\pi(2a + t)}{2d} \left[\cos \frac{n\pi t}{2d} \left(1 - \epsilon^{-\frac{n\pi\ell}{d}} \cosh \frac{n\pi x}{d} \right) \right.$$

$$\left. + \sin \frac{n\pi t}{2d} \epsilon^{-\frac{n\pi\ell}{d}} \cosh \frac{n\pi x}{d} \right] \tag{2.12}$$

Equation (2.12) is next integrated in the xy plane:

$$J = \int_{-\ell}^{+\ell} J_1(x,a)dx - \int_{+\ell}^{-\ell} J_1(x,a + t)dx + \int_{a}^{d-b} J_1(\ell,y)dy - \int_{d-b}^{a} J_1(-\ell,y)dy \tag{2.13}$$

The final answer was found to be

$$J = \frac{8d^2}{n^2\pi^2} \sin^2 \frac{n\pi(2a + t)}{2d} \left[\frac{2\ell n\pi}{d} \cos^2 \frac{n\pi t}{2d} + \sin \frac{n\pi t}{d} - \cos \frac{n\pi t}{d} \right.$$

$$\left. + \left(1 - \sin \frac{n\pi t}{d} \right) \epsilon^{-\frac{2n\pi\ell}{d}} \right] \tag{2.14}$$

The upper bound of Z_0 is then given by:

$$Z_0^U = \sum_{n=1}^{\infty} \left(\frac{2d^2 Z_0}{\kappa\pi^3(t + 2\ell)^2} \sin^2 \frac{n\pi(2a + t)}{2d} \right) \left(\frac{2\ell\pi}{n^2 d} \cos^2 \frac{n\pi t}{2d} \right.$$

$$\left. + \frac{1}{n^3} \left(\sin \frac{n\pi t}{d} - \cos \frac{n\pi t}{d} \right) + \frac{1}{n^3} \left(1 - \sin \frac{n\pi t}{d} \right) \epsilon^{-\frac{2n\pi\ell}{d}} \right) \tag{2.15}$$

The limiting case of a thin-center conductor is described by:

$$Z_0^U = \sum_{n=1}^{\infty} \left(\frac{d^2 Z_0}{2\kappa\pi^3\ell^2 n^3} \sin^2 \frac{n\pi a}{d} \right) \left(\frac{2n\pi\ell}{d} - 1 + \epsilon^{-\frac{2n\pi\ell}{d}} \right) \tag{2.16}$$

Equations (1.21b) and (2.16) have been evaluated for different geometries. The answers are satisfactory for many applications. The arithmetic means of Z_0^L and Z_0^U is within eight per cent of the actual value for all geometries tested. For the low impedance lines of interest here this deviation is halved. This results in designable line impedances with uncertainties of the same order as those in the resistors.

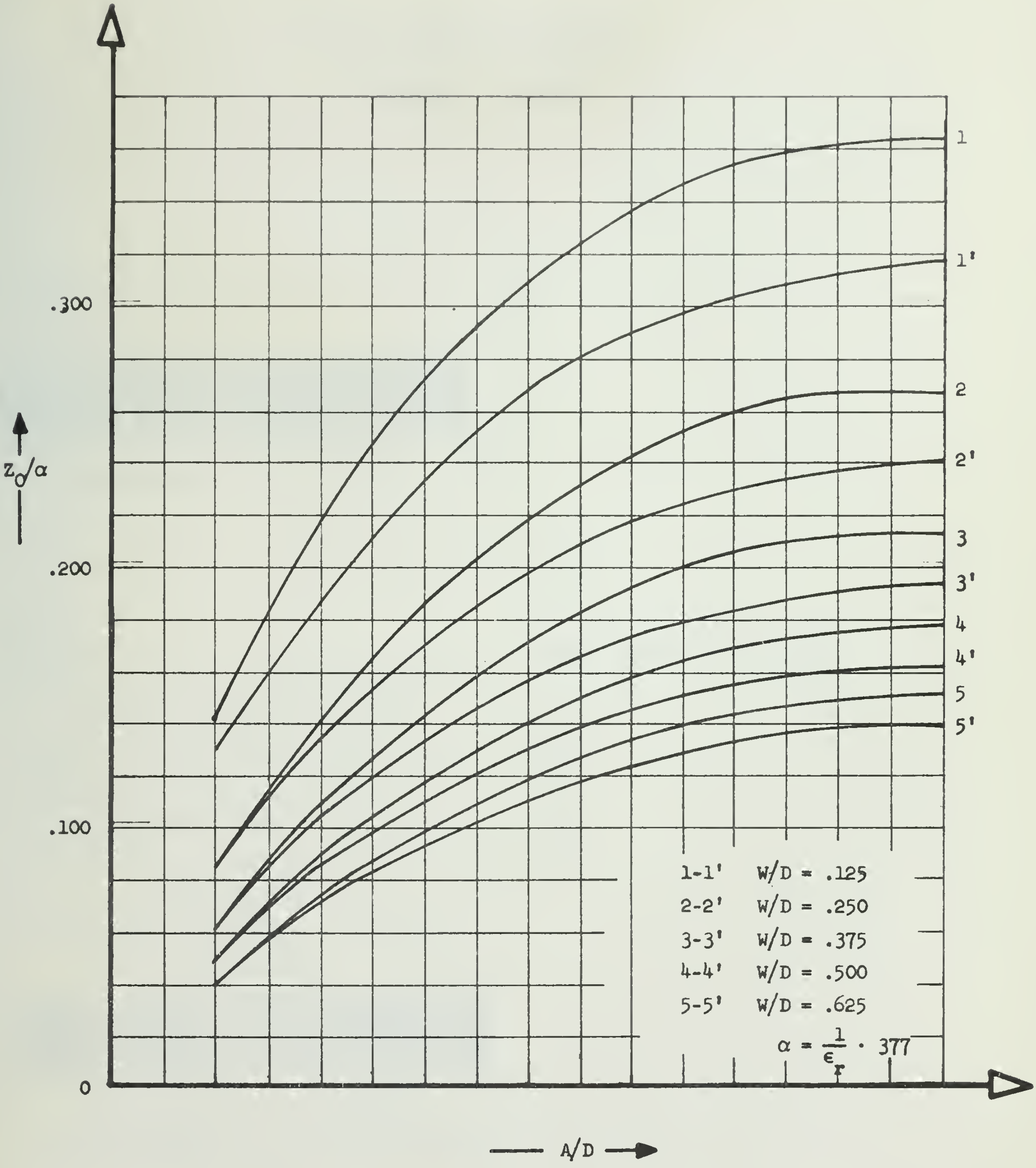

Figure 3. Upper and Lower Bounds of Characteristic Impedance for Thin-Center Conductor.